

# Diagrams and Procedures for Partition of Variation

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processed with `vegan 2.7-0` in R version 4.4.1 (2024-06-14) on August 31, 2024

Diagrams describing the partitions of variation of a response data table by two (Fig. 1), three (Fig. 2) and four tables (Fig. 3) of explanatory variables. The fraction names [a] to [p] in the output of `varpart` function follow the notation in these Venn diagrams, and the diagrams were produced using the `showvarparts` function.

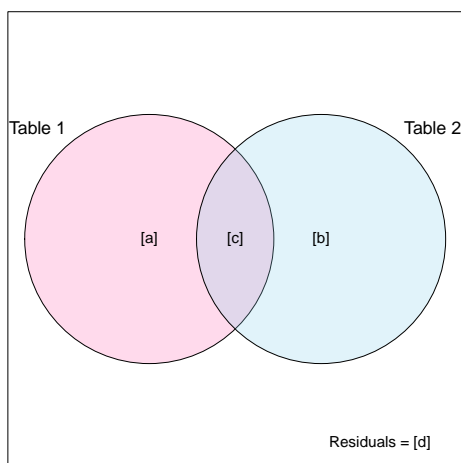


Figure 1: 3 regression/canonical analyses and 3 subtraction equations are needed to estimate the 4 ( $= 2^2$ ) fractions. [a] and [c] can be tested for significance (3 canonical analyses per permutation). Fraction [b] cannot be tested singly.

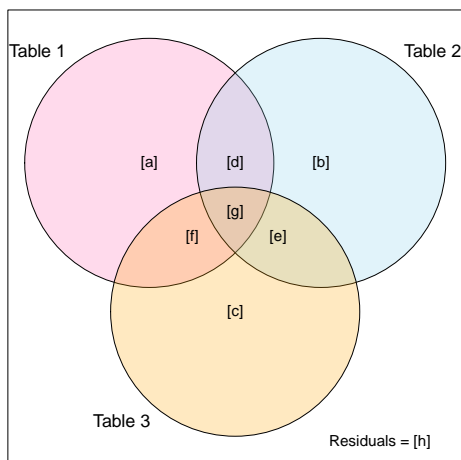


Figure 2: 7 regression/canonical analyses and 10 subtraction equations are needed to estimate the  $8 (= 2^3)$  fractions. [a] to [c] and subsets containing [a] to [c] can be tested for significance (4 canonical analyses per permutation to test [a] to [c]). Fractions [d] to [g] cannot be tested singly.

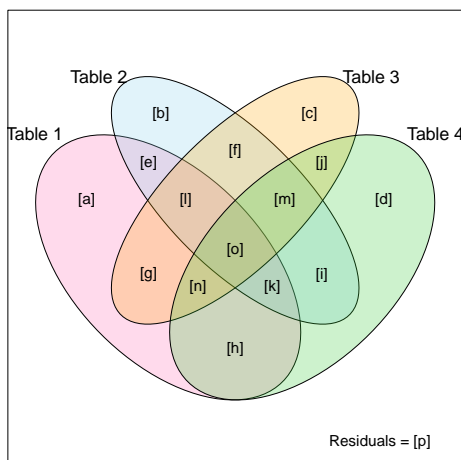


Figure 3: 15 regression/canonical analyses and 27 subtraction equations are needed to estimate the  $16 (= 2^4)$  fractions. [a] to [d] and subsets containing [a] to [d] can be tested for significance (5 canonical analyses per permutation to test [a] to [d]). Fractions [e] to [o] cannot be tested singly.

Variation partitioning for two explanatory data tables --  
 Table 1 with m1 explanatory variables, Table 2 with m2 explanatory variables  
 Number of fractions: 4, called [a] ... [d]  
 √ indicates the 3 regression or canonical analyses that have to be computed.  
 # Partial canonical analyses are only computed if tests of significance or biplots are needed.

Compute	Fitted	Residuals	Derived fractions	Degrees of freedom, numerator of F
√ Y.1	[a+b]	[c+d] (1)		df(a+b) = m1
√ Y.2	[b+c]	[a+d] (2)		df(b+c) = m2
√ Y.1,2	[a+b+c]	[d] (3)		df(a+b+c) = m3 ≤ m1+m2 (there may be collinearity)
# Y.1 2	[a]	[d]		df(a) = m3-m2
# Y.2 1	[c]	[d]		df(c) = m3-m1

Partial analyses  
 controlling for 1 table X (4) [a] = [a+b+c] - [b+c] df(a) = m3-m2\*  
 (5) [c] = [a+b+c] - [a+b] df(c) = m3-m1\*  
 (6) [b] = [a+b] + [b+c] - [a+b+c] df(b) = m1+m2-(m1+m2) = 0  
 (7) [d] = residuals = 1 - [a+b+c] df2(d) = n-1-m3 for denominator of F  
 \* Calculation of d.f. for difference between nested models: see Sokal & Rohlf (1981, 1995) equation 16.14.

Tests of significance --

$$F(a+b) = ([a+b]/m1)/([c+d]/(n-1-m1))$$

$$F(b+c) = ([b+c]/m2)/([a+d]/(n-1-m2))$$

$$F(a+b+c) = ([a+b+c]/m3)/([d]/(n-1-m3))$$

$$F(a) = ([a]/(m3-m2))/([d]/(n-1-m3))$$

$$F(c) = ([c]/(m3-m1))/([d]/(n-1-m3))$$

The only testable fractions are those that can be obtained directly by regression or canonical analysis.  
 The non-testable fraction is [b]. That fraction cannot be obtained directly by regression or canonical analysis.

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Variation partitioning for three explanatory data tables --  
 Table 1 with m1 explanatory variables, Table 2 with m2 explanatory variables, Table3 with m3 explanatory variables  
 Number of fractions: 8, called [a] ... [h]  
 √ indicates the 7 regression or canonical analyses that have to be computed.  
 # Partial canonical analyses are only computed if tests of significance or biplots are needed.

Compute	Fitted	Residuals	Derived fractions	Degrees of freedom, numerator of F
Direct canonical analysis				
√ Y.1	[a+d+f+g]	[b+c+e+h] (1)		df(a+d+f+g) = m1
√ Y.2	[b+d+e+g]	[a+c+f+h] (2)		df(b+d+e+g) = m2
√ Y.3	[c+e+f+g]	[a+b+d+h] (3)		df(c+e+f+g) = m3
√ Y.1,2	[a+b+d+e+f+g]	[c+h] (4)		df(a+b+d+e+f+g) = m4 ≤ m1+m2 (collinearity?)
√ Y.1,3	[a+c+d+e+f+g]	[b+h] (5)		df(a+c+d+e+f+g) = m5 ≤ m1+m3 (collinearity?)
√ Y.2,3	[b+c+d+e+f+g]	[a+h] (6)		df(b+c+d+e+f+g) = m6 ≤ m2+m3 (collinearity?)
√ Y.1,2,3	[a+b+c+d+e+f+g]	[h] (7)		df(a+b+c+d+e+f+g) = m7 ≤ m1+m2+m3 (collinearity?)
# Y.1 2	[a+f]	[c+h]		df(a+f) = m4-m2
# Y.1 3	[a+d]	[b+h]		df(a+d) = m5-m3
# Y.2 1	[b+e]	[c+h]		df(b+e) = m4-m1
# Y.2 3	[b+d]	[a+h]		df(b+d) = m6-m3
# Y.3 1	[c+e]	[b+h]		df(c+e) = m5-m1
# Y.3 2	[c+f]	[a+h]		df(c+f) = m6-m2
# Y.1 2,3	[a]	[h]		df(a) = m7-m6
# Y.2 1,3	[b]	[h]		df(b) = m7-m5
# Y.3 1,2	[c]	[h]		df(c) = m7-m4

Partial analyses  
 controlling for two tables X (8) [a] = [a+b+c+d+e+f+g] - [b+c+d+e+f+g] df(a) = m7-m6  
 (9) [b] = [a+b+c+d+e+f+g] - [a+c+d+e+f+g] df(b) = m7-m5  
 (10) [c] = [a+b+c+d+e+f+g] - [a+b+d+e+f+g] df(c) = m7-m4

controlling for one table X (11) [a+d] = [a+c+d+e+f+g] - [c+e+f+g] df(a+d) = m5-m3  
 (12) [a+f] = [a+b+d+e+f+g] - [b+d+e+g] df(a+f) = m4-m2  
 (13) [b+d] = [b+c+d+e+f+g] - [c+e+f+g] df(b+d) = m6-m3  
 (14) [b+e] = [a+b+d+e+f+g] - [a+d+f+g] df(b+e) = m4-m1  
 (15) [c+e] = [a+c+d+e+f+g] - [a+d+f+g] df(c+e) = m5-m1  
 (16) [c+f] = [b+c+d+e+f+g] - [b+d+e+g] df(c+f) = m6-m2

Fractions estimated  
 by subtraction (17) [d] = [a+d] - [a] df(d) = m1-m1 = 0  
 (18) [e] = [b+e] - [b] df(e) = m2-m2 = 0  
 (cannot be tested) (19) [f] = [c+f] - [c] df(f) = m3-m3 = 0  
 (20) [g] = [a+b+c+d+e+f+g] - [a+d] - [b+e] - [c+f] df(g) = (m1+m2+m3)-m1-m2-m3 = 0  
 or [g] = [a+d+f+g] - [a] - [d] - [f] df(g) = m1-m1-0-0 = 0  
 (21) [h] = residuals = 1 - [a+b+c+d+e+f+g] df2(h) = n-1-m7 for denominator of F

Tests of significance --

$$F(a+d+f+g) = ([a+d+f+g]/m1)/([b+c+e+h]/(n-1-m1))$$

$$F(b+d+e+g) = ([b+d+e+g]/m2)/([a+c+f+h]/(n-1-m2))$$

$$F(c+e+f+g) = ([c+e+f+g]/m3)/([a+b+d+h]/(n-1-m3))$$

$$F(a+b+d+e+f+g) = ([a+b+d+e+f+g]/m4)/([c+h]/(n-1-m4))$$

$$F(a+c+d+e+f+g) = ([a+c+d+e+f+g]/m5)/([b+h]/(n-1-m5))$$

$$F(b+c+d+e+f+g) = ([b+c+d+e+f+g]/m6)/([a+h]/(n-1-m6))$$

$$F(a+b+c+d+e+f+g) = ([a+b+c+d+e+f+g]/m7)/([h]/(n-1-m7))$$

$$F(a) = ([a]/(m7-m6))/([h]/(n-1-m7))$$

$$F(b) = ([b]/(m7-m5))/([h]/(n-1-m7))$$

$$F(c) = ([c]/(m7-m4))/([h]/(n-1-m7))$$

$$F(a+d) = ([a+d]/(m5-m3))/([b+h]/(n-1-m5))$$

$$F(a+f) = ([a+f]/(m4-m2))/([c+h]/(n-1-m4))$$

$$F(b+d) = ([b+d]/(m6-m3))/([a+h]/(n-1-m6))$$

$$F(b+e) = ([b+e]/(m4-m1))/([c+h]/(n-1-m4))$$

$$F(c+e) = ([c+e]/(m5-m1))/([b+h]/(n-1-m5))$$

$$F(c+f) = ([c+f]/(m6-m2))/([a+h]/(n-1-m6))$$

The only testable fractions are those that can be obtained directly by regression or canonical analysis.

Variation partitioning for four explanatory data tables --  
 Table 1 with m1 variables, Table 2 with m2 variables, Table3 with m3 variables, Table4 with m4 variables  
 Number of fractions: 16, called [a] ... [p].  
 √ indicates the 15 regression or canonical analyses that have to be computed.

Compute	Fitted	Residuals	Derived fractions	Degrees of freedom
<b>Direct canonical analysis</b>				
√ Y.1	[a+e+g+h+k+l+n+o]	[b+c+d+f+i+j+m+p]	(1)	df(a+e+g+h+k+l+n+o) = m1
√ Y.2	[b+e+f+i+k+l+m+o]	[a+c+d+g+h+j+n+p]	(2)	df(b+e+f+i+k+l+m+o) = m2
√ Y.3	[c+f+g+j+l+m+n+o]	[a+b+d+e+h+i+k+p]	(3)	df(c+f+g+j+l+m+n+o) = m3
√ Y.4	[d+h+i+j+k+m+n+o]	[a+b+c+e+f+g+l+p]	(4)	df(d+h+i+j+k+m+n+o) = m4
√ Y.1,2	[a+b+e+f+g+h+i+k+l+m+n+o]	[c+d+j+p]	(5)	df(a+b+e+f+g+h+i+k+l+m+n+o) = m5 ≤ m1+m2
√ Y.1,3	[a+c+e+f+g+h+j+k+l+m+n+o]	[b+d+i+p]	(6)	df(a+c+e+f+g+h+j+k+l+m+n+o) = m6 ≤ m1+m3
√ Y.1,4	[a+d+e+g+h+i+j+k+l+m+n+o]	[b+c+f+p]	(7)	df(a+d+e+g+h+i+j+k+l+m+n+o) = m7 ≤ m1+m4
√ Y.2,3	[b+c+e+f+g+i+j+k+l+m+n+o]	[a+d+h+p]	(8)	df(b+c+e+f+g+i+j+k+l+m+n+o) = m8 ≤ m2+m3
√ Y.2,4	[b+d+e+f+h+i+j+k+l+m+n+o]	[a+c+g+p]	(9)	df(b+d+e+f+h+i+j+k+l+m+n+o) = m9 ≤ m2+m4
√ Y.3,4	[c+d+f+g+h+i+j+k+l+m+n+o]	[a+b+e+p]	(10)	df(c+d+f+g+h+i+j+k+l+m+n+o) = m10 ≤ m3+m4
√ Y.1,2,3	[a+b+c+e+f+g+h+i+j+k+l+m+n+o]	[d+p]	(11)	df(a+b+c+e+f+g+h+i+j+k+l+m+n+o) = m11 ≤ m1+m2+m3
√ Y.1,2,4	[a+b+d+e+f+g+h+i+j+k+l+m+n+o]	[c+p]	(12)	df(a+b+d+e+f+g+h+i+j+k+l+m+n+o) = m12 ≤ m1+m2+m4
√ Y.1,3,4	[a+c+d+e+f+g+h+i+j+k+l+m+n+o]	[b+p]	(13)	df(a+c+d+e+f+g+h+i+j+k+l+m+n+o) = m13 ≤ m1+m3+m4
√ Y.2,3,4	[b+c+d+e+f+g+h+i+j+k+l+m+n+o]	[a+p]	(14)	df(b+c+d+e+f+g+h+i+j+k+l+m+n+o) = m14 ≤ m2+m3+m4
√ Y.1,2,3,4	[a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]	[p]	(15)	df(a+b+c+d+e+f+g+h+i+j+k+l+m+n+o) = m15 ≤ m1+m2+m3+m4
<b>Partial analyses</b>				
controlling for one table X				
(16)	[a+g+h+n]	[a+b+e+f+g+h+i+k+l+m+n+o]	- [b+e+f+i+k+l+m+o]	df(a+g+h+n) = m5 - m2
(17)	[a+e+h+k]	[a+c+e+f+g+h+j+k+l+m+n+o]	- [c+f+g+j+l+m+n+o]	df(a+e+h+k) = m6 - m3
(18)	[a+e+g+l]	[a+d+e+g+h+i+j+k+l+m+n+o]	- [d+h+i+j+k+m+n+o]	df(a+e+g+l) = m7 - m4
(19)	[b+f+i+m]	[a+b+e+f+g+h+i+k+l+m+n+o]	- [a+e+g+h+k+l+n+o]	df(b+f+i+m) = m5 - m1
(20)	[b+e+i+k]	[b+c+e+f+g+i+j+k+l+m+n+o]	- [c+f+g+j+l+m+n+o]	df(b+e+i+k) = m8 - m3
(21)	[b+e+f+l]	[b+d+e+f+h+i+j+k+l+m+n+o]	- [d+h+i+j+k+m+n+o]	df(b+e+f+l) = m9 - m4
(22)	[c+f+j+m]	[a+c+e+f+g+h+j+k+l+m+n+o]	- [a+e+g+h+k+l+n+o]	df(a) = m6 - m1
(23)	[c+g+j+n]	[b+c+e+f+g+i+j+k+l+m+n+o]	- [b+e+f+i+k+l+m+o]	df(a) = m8 - m2
(24)	[c+f+g+l]	[c+d+f+g+h+i+j+k+l+m+n+o]	- [d+h+i+j+k+m+n+o]	df(a) = m10 - m4
(25)	[d+i+j+m]	[a+d+e+g+h+i+j+k+l+m+n+o]	- [a+e+g+h+k+l+n+o]	df(a) = m7 - m1
(26)	[d+h+j+n]	[b+d+e+f+h+i+j+k+l+m+n+o]	- [b+e+f+i+k+l+m+o]	df(a) = m9 - m2
(27)	[d+h+i+k]	[c+d+f+g+h+i+j+k+l+m+n+o]	- [c+f+g+j+l+m+n+o]	df(a) = m10 - m3
controlling for two tables X				
(28)	[a+e]	[a+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [c+d+f+g+h+i+j+k+l+m+n+o]	df(a+e) = m13 - m10
(29)	[a+g]	[a+b+d+e+f+g+h+i+j+k+l+m+n+o]	- [b+d+e+f+h+i+j+k+l+m+n+o]	df(a+g) = m12 - m9
(30)	[a+h]	[a+b+c+e+f+g+h+i+j+k+l+m+n+o]	- [b+c+e+f+g+i+j+k+l+m+n+o]	df(a+h) = m11 - m8
(31)	[b+e]	[b+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [c+d+f+g+h+i+j+k+l+m+n+o]	df(b+e) = m14 - m10
(32)	[b+f]	[a+b+d+e+f+g+h+i+j+k+l+m+n+o]	- [a+d+e+g+h+i+j+k+l+m+n+o]	df(b+f) = m12 - m7
(33)	[b+i]	[a+b+c+e+f+g+h+i+j+k+l+m+n+o]	- [a+c+e+f+g+h+j+k+l+m+n+o]	df(b+i) = m11 - m6
(34)	[c+f]	[a+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [a+d+e+g+h+i+j+k+l+m+n+o]	df(c+f) = m13 - m7
(35)	[c+g]	[b+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [b+d+e+f+h+i+j+k+l+m+n+o]	df(c+g) = m14 - m9
(36)	[c+j]	[a+b+c+e+f+g+h+i+j+k+l+m+n+o]	- [a+b+e+f+g+h+i+k+l+m+n+o]	df(c+j) = m11 - m5
(37)	[d+h]	[b+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [b+c+e+f+g+i+j+k+l+m+n+o]	df(d+h) = m14 - m8
(38)	[d+i]	[a+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [a+c+e+f+g+h+j+k+l+m+n+o]	df(d+i) = m13 - m6
(39)	[d+j]	[a+b+d+e+f+g+h+i+j+k+l+m+n+o]	- [a+b+e+f+g+h+i+k+l+m+n+o]	df(d+j) = m12 - m5
controlling for three tables X				
(40)	[a]	[a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [b+c+d+e+f+g+h+i+j+k+l+m+n+o]	df(a) = m15 - m14
(41)	[b]	[a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [a+c+d+e+f+g+h+i+j+k+l+m+n+o]	df(b) = m15 - m13
(42)	[c]	[a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [a+b+d+e+f+g+h+i+j+k+l+m+n+o]	df(c) = m15 - m12
(43)	[d]	[a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]	- [a+b+c+e+f+g+h+i+j+k+l+m+n+o]	df(d) = m15 - m11
Fractions estimated by subtraction (cannot be tested)				
(44)	[e]	[a+e]	- [a]	df(e) = m1-m1 = 0
(45)	[f]	[b+f]	- [b]	df(f) = m2-m2 = 0
(46)	[g]	[a+g]	- [a]	df(g) = m1-m1 = 0
(47)	[h]	[a+h]	- [a]	df(h) = m1-m1 = 0
(48)	[i]	[b+i]	- [b]	df(i) = m2-m2 = 0
(49)	[j]	[c+j]	- [c]	df(j) = m3-m3 = 0
(50)	[k]	[a+e+h+k]	- [a+e] - [h]	df(k) = m1-m1-0 = 0
(51)	[l]	[a+e+g+l]	- [a+e] - [g]	df(l) = m1-m1-0 = 0
(52)	[m]	[b+f+i+m]	- [b+f] - [i]	df(m) = m2-m2-0 = 0
(53)	[n]	[a+g+h+n]	- [a+g] - [h]	df(n) = m1-m1-0 = 0
(54)	[o]	[a+e+g+h+k+l+n+o]	- [a+e+h+k] - [g] - [l] - [n]	df(o) = m1-m1-0-0-0 = 0
(55)	[p]	residuals	= 1 - [a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]	df2(p) = n-1-m15

Tests of significance --

$F(a+e+g+h+k+l+n+o) = ([a+e+g+h+k+l+n+o]/m1)/([b+c+d+f+i+j+m+p]/(n-1-m1))$   
 $F(b+e+f+i+k+l+m+o) = ([b+e+f+i+k+l+m+o]/m2)/([a+c+d+g+h+j+n+p]/(n-1-m2))$   
 $F(c+f+g+j+l+m+n+o) = ([c+f+g+j+l+m+n+o]/m3)/([a+b+d+e+h+i+k+p]/(n-1-m3))$   
 $F(d+h+i+j+k+m+n+o) = ([d+h+i+j+k+m+n+o]/m4)/([a+b+c+e+f+g+l+p]/(n-1-m4))$   
 $F(a+b+e+f+g+h+i+k+l+m+n+o) = ([a+b+e+f+g+h+i+k+l+m+n+o]/m5)/([c+d+j+p]/(n-1-m5))$   
 $F(a+c+e+f+g+h+j+k+l+m+n+o) = ([a+c+e+f+g+h+j+k+l+m+n+o]/m6)/([b+d+i+p]/(n-1-m6))$   
 $F(a+d+e+g+h+i+j+k+l+m+n+o) = ([a+d+e+g+h+i+j+k+l+m+n+o]/m7)/([b+c+f+p]/(n-1-m7))$   
 $F(b+c+e+f+g+i+j+k+l+m+n+o) = ([b+c+e+f+g+i+j+k+l+m+n+o]/m8)/([a+d+h+p]/(n-1-m8))$   
 $F(b+d+e+f+h+i+j+k+l+m+n+o) = ([b+d+e+f+h+i+j+k+l+m+n+o]/m9)/([a+c+g+p]/(n-1-m9))$   
 $F(c+d+f+g+h+i+j+k+l+m+n+o) = ([c+d+f+g+h+i+j+k+l+m+n+o]/m10)/([a+b+e+p]/(n-1-m10))$   
 $F(a+b+c+e+f+g+h+i+j+k+l+m+n+o) = ([a+b+c+e+f+g+h+i+j+k+l+m+n+o]/m11)/([d+p]/(n-1-m11))$   
 $F(a+b+d+e+f+g+h+i+j+k+l+m+n+o) = ([a+b+d+e+f+g+h+i+j+k+l+m+n+o]/m12)/([c+p]/(n-1-m12))$   
 $F(a+c+d+e+f+g+h+i+j+k+l+m+n+o) = ([a+c+d+e+f+g+h+i+j+k+l+m+n+o]/m13)/([b+p]/(n-1-m13))$   
 $F(b+c+d+e+f+g+h+i+j+k+l+m+n+o) = ([b+c+d+e+f+g+h+i+j+k+l+m+n+o]/m14)/([a+p]/(n-1-m14))$   
 $F(a+b+c+d+e+f+g+h+i+j+k+l+m+n+o) = ([a+b+c+d+e+f+g+h+i+j+k+l+m+n+o]/m15)/([p]/(n-1-m15))$

$F(a+g+h+n) = ([a+g+h+n]/(m5-m2))/([c+d+j+p]/(n-1-m5))$   
For the other fractions controlling for one table X, the F-statistics are constructed in the same way

$F(a+e) = ([a+e]/(m13-m10))/([b+p]/(n-1-m13))$   
For the other fractions controlling for two tables X, the F-statistics are constructed in the same way

Fractions controlling for three tables X:  
 $F(a) = ([a]/(m15-m14))/([p]/(n-1-m15))$   
 $F(b) = ([b]/(m15-m13))/([p]/(n-1-m15))$   
 $F(c) = ([c]/(m15-m12))/([p]/(n-1-m15))$   
 $F(d) = ([d]/(m15-m11))/([p]/(n-1-m15))$

Other fractions combining elementary fractions [a] to [o] can be calculated, but cannot be tested because they cannot be obtained by regression.

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